Polyadic Quantification

without

Identity

Exercises and Solutions

#1 $\{\forall x(F(x) \Rightarrow \exists y\, G(x,y))\} \vdash \forall x \exists y(F(x) \Rightarrow G(x,y))$

1	$\forall x(F(x) \Rightarrow \exists y\, G(x,y))$ /∴ $\forall x \exists y(F(x) \Rightarrow G(x,y))$	
2	$\neg \forall x \exists y(F(x) \Rightarrow G(x,y))$ /∴ $\bot$	
3	$\exists x \forall y \neg(F(x) \Rightarrow G(x,y))$	QN, 2
4	$\exists x \forall y(F(x) \wedge \neg G(x,y))$	SL
5	$\exists x$	EC
6	$F(x) \Rightarrow \exists y\, G(x,y)$	EC, 1
7	$\forall y(F(x) \wedge \neg G(x,y))$	EC, 4
8	$\forall y\, F(x) \wedge \forall y \neg G(x,y)$	QD, 7
9	$\forall y\, F(x)$	$\wedge$E, 8
10	$F(x)$	VQ 9
11	$\forall y \neg G(x,y)$	$\wedge$E, 8
12	$\exists y\, G(x,y)$	$\Rightarrow$E, 6, 10
13	$\exists y$	EC
14	$\neg G(x,y)$	EC 11
15	$G(x,y)$	EC 12
16	$\bot$	14, 15
17	$\bot$	
18	$\bot$	
19	$\forall x \exists y(F(x) \Rightarrow G(x,y))$	$\neg$E, 2-18

#2 $\{\forall x P(x) \Rightarrow \forall y\, Q(y)\} \vdash \exists x \forall y (P(x) \Rightarrow Q(y))$

1	$\forall x P(x) \Rightarrow \forall y Q(y)$ $\quad$ /∴ $\exists x \forall y (P(x) \Rightarrow Q(y))$	
2	$\neg \exists x \forall y (P(x) \Rightarrow Q(y))$ $\quad$ /∴ $\perp$	
3	$\forall x \exists y \neg (P(x) \Rightarrow Q(y))$	QN2
4	$\forall x \exists y (P(x) \wedge \neg Q(y))$	SL
5	$\forall x$	UC
6	$\exists y (P(x) \wedge \neg Q(y))$	UC 4
7	$\exists y$	EC
8	$P(x) \wedge \neg Q(y)$	EC 6
9	$P(x)$	$\wedge$E 8
10	$\neg Q(y)$	$\wedge$E 8
11	$\exists y \neg Q(y)$	EC 10
12	$P(x)$	EVDQ 9
13	$\exists y \neg Q(y)$	EVDQ 11
14	$\forall x P(x)$	UDC 12
15	$\exists y \neg Q(y)$	UVDC 13
16	$\forall y Q(y)$	$\Rightarrow$E, 1, 14
17	$\neg \forall y Q(y)$	QN 15
18	$\perp$	$\perp$ 16,17
19	$\exists x \forall y (P(x) \Rightarrow Q(y))$	$\neg$E, 2–18

A different method that does not use the RAA method.

1	$\forall x\, P(x) \Rightarrow \forall y\, Q(y) \quad /\therefore\ \exists x\, \forall y\, (P(x) \Rightarrow Q(y))$	
2	$\neg \forall x\, P(x) \vee \forall y\, Q(y)$	
3	$\quad \neg \forall x\, P(x)$	
4	$\quad \exists x\, \neg P(x)$	QN 3
5	$\quad\quad \exists x$	EC
6	$\quad\quad \neg P(x)$	EC 4
7	$\quad\quad\quad \forall y$	UC
8	$\quad\quad\quad \neg P(x)$	Imp. 6
9	$\quad\quad\quad \neg P(x) \vee Q(y)$	$\vee$I, 8
10	$\quad\quad \forall y\, (\neg P(x) \vee Q(y))$	UDC 7–9
11	$\quad\quad \forall y\, (P(x) \Rightarrow Q(y))$	SL 10
12	$\quad \exists x\, \forall y\, (P(x) \Rightarrow Q(y))$	EDC 5–11
13	$\quad \forall y\, Q(y)$	
14	$\quad\quad \exists x$	EC
15	$\quad\quad\quad \forall y$	UC
16	$\quad\quad\quad Q(y)$	UC 13
17	$\quad\quad\quad \neg P(x) \vee Q(y)$	$\vee$I, 16
18	$\quad\quad\quad (P(x) \Rightarrow Q(y))$	SL 17
19	$\quad\quad \forall y\, (P(x) \Rightarrow Q(y))$	UDC 15–18
20	$\quad \exists x\, \forall y\, (P(x) \Rightarrow Q(y))$	EDC 14–19
21	$\exists x\, \forall y\, (P(x) \Rightarrow Q(y))$	$\vee$E, 2, 3–12, 13–20

#3 $\{\exists x\, \forall y\, (P(x) \Rightarrow Q(y))\} \vdash \forall x\, P(x) \Rightarrow \forall y\, Q(y)$

1	$\exists x\, \forall y\, (P(x) \Rightarrow Q(y))$ /∴ $\forall x\, P(x) \Rightarrow \forall y\, Q(y)$	
2	$\forall x\, P(x)$ /∴ $\forall y\, Q(y)$	
3	$\exists x$	EC
4	$\forall y\, (P(x) \Rightarrow Q(y))$	EC 1
5	$P(x)$	EC 2
6	$\forall y$	UC
7	$P(x) \Rightarrow Q(y)$	UC 4
8	$P(x)$	Imp 5 allowed; $y \notin FV[P(x)]$
9	$Q(y)$	$\Rightarrow$E, 7, 8
10	$\forall y\, Q(y)$	UDC 9
11	$\forall y\, Q(y)$	EVDC 10
12	$\forall x\, P(x) \Rightarrow \forall y\, Q(y)$	$\Rightarrow$I, 2 - 11

We have shown:

$$\forall x\, P(x) \Rightarrow \forall y\, Q(y) \dashv\vdash \exists x\, \forall y\, (P(x) \Rightarrow Q(y)),$$

that is, the following is a theorem:

$$\phi \vdash (\forall x\, P(x) \Rightarrow \forall y\, Q(y)) \Leftrightarrow \exists x\, \forall y\, (P(x) \Rightarrow Q(y))$$

#4 $\{\exists x\, P(x) \Rightarrow \exists y\, Q(y)\} \vdash \forall x\, \exists y\, (P(x) \Rightarrow Q(y))$

1	$\exists x\, P(x) \Rightarrow \exists y\, Q(y)$ /∴ $\forall x\, \exists y\, (P(x) \Rightarrow Q(y))$	
2	$\neg \forall x\, \exists y\, (P(x) \Rightarrow Q(y))$ /∴ $\perp$	
3	$\exists x\, \forall y\, \neg (P(x) \Rightarrow Q(y))$	QN 2
4	$\exists x\, \forall y\, (P(x) \wedge \neg Q(y))$	SL
5	$\exists x$	EC
6	$\forall y\, (P(x) \wedge \neg Q(y))$	EC 4
7	$\forall y$	UC
8	$P(x) \wedge \neg Q(y)$	UC 6
9	$P(x)$	$\wedge$E, 8
10	$\exists x\, P(x)$	EG, 9
11	$\exists y\, Q(y)$	$\Rightarrow$E, 1, 10
12	$\neg Q(y)$	$\wedge$E, 8
13	$\forall y\, \neg Q(y)$	UDC 12
14	$\exists y\, Q(y)$	VUDC 11
15	$\neg \exists y\, Q(y)$	QN 13
16	$\perp$	14, 15
17	$\perp$	
18	$\forall x\, \exists y\, (P(x) \Rightarrow Q(y))$	$\neg$E, 2–17

#5 $\{\forall x \exists y\, (P(x) \Rightarrow Q(y))\} \vdash \exists x\, P(x) \Rightarrow \exists y\, Q(y)$

1	$\forall x \exists y\, (P(x) \Rightarrow Q(y))$ $\quad$ /∴ $\exists x\, P(x) \Rightarrow \exists y\, Q(y)$	
2	$\quad$ $\exists x\, P(x)$ $\quad$ /∴ $\exists y\, Q(y)$	
3	$\quad\quad$ $\exists x$	EC
4	$\quad\quad$ $P(x)$	EC 2
5	$\quad\quad$ $\exists y\, (P(x) \Rightarrow Q(y))$	EC 1
6	$\quad\quad\quad$ $\exists y$	EC
7	$\quad\quad\quad$ $(P(x) \Rightarrow Q(y))$	EC 5
8	$\quad\quad\quad$ $P(x)$	Imp 4
9	$\quad\quad\quad$ $Q(y)$	$\Rightarrow$E, 7, 8
10	$\quad\quad$ $\exists y\, Q(y)$	EDC 9
11	$\quad$ $\exists y\, Q(y)$	VEDC 10
12	$\exists x\, P(x) \Rightarrow \exists y\, Q(y)$	$\Rightarrow$I, 2-11

$$\exists x\, P(x) \Rightarrow \exists y\, Q(y) \vdash\dashv \forall x \exists y\, (P(x) \Rightarrow Q(y))$$

$$\phi \vdash (\exists_x P(x) \Rightarrow \exists_y Q(y)) \Leftrightarrow \forall x \exists y\, (P(x) \Rightarrow Q(y))$$

#6 $\{\forall x \exists y (P(x) \Leftrightarrow \neg P(y))\} \vdash \exists x P(x) \wedge \exists x \neg P(x)$

1	$\forall x \exists y (P(x) \Leftrightarrow \neg P(y))$ $/\therefore \exists x P(x) \wedge \exists x \neg P(x)$	
2	$\neg \exists x P(x)$ $/\therefore \perp$	
3	$\forall x \neg P(x)$	QN
4	$\forall x$	UC
5	$\exists y (P(x) \Leftrightarrow \neg P(y))$	UC 1
6	$\neg P(x)$	UC 3
7	$\exists y$	EC
8	$P(x) \Leftrightarrow \neg P(y)$	EC 5
9	$\neg P(x)$	Imp 6
10	$P(y)$	SL
11	$\exists y P(y)$	EDC
12	$\exists y P(y)$	VUDC 4 - 11
13	$\exists x P(x)$	RDV
14	$\perp$	2, 13
15	$\exists x P(x)$	$\neg$E, 2-14
16	$\neg \exists x \neg P(x)$ $/\therefore \perp$	
17	$\forall x P(x)$	QN 16
18	$\forall x$	UC
19	$\exists y (P(x) \Leftrightarrow \neg P(y))$	UC 1
20	$P(x)$	UC 17
21	$\exists y$	EC
22	$P(x) \Leftrightarrow \neg P(y)$	EC 19
23	$P(x)$	Imp 20
24	$\neg P(y)$	SL
25	$\exists y \neg P(y)$	EDC 21-24
26	$\exists y \neg P(y)$	VEDC 18-25
27	$\neg \forall y P(y)$	QN 26
28	$\neg \forall x P(x)$	RDV 27
29	$\perp$	17, 28
30	$\exists x \neg P(x)$	$\neg$E 16-29
31	$\exists x P(x) \wedge \exists x \neg P(x)$	$\wedge$I, 15, 30

Different method proving #6

1	$\forall x \exists y (P(x) \Leftrightarrow \neg P(y))$ $\quad$ /∴ $\exists x P(x) \wedge \exists x \neg P(x)$	
2	$\quad \underline{\forall x}$	UC
3	$\quad\quad \underline{\exists y}$	EC
4	$\quad\quad P(x) \Leftrightarrow \neg P(y)$	UC+EC, 1
5	$\quad\quad\quad \underline{\neg \exists x P(x)}$ $\quad$ /∴ $\perp$	
6	$\quad\quad\quad \forall x \neg P(x)$	QN 5
7	$\quad\quad\quad \neg P(y)$	UI, 6, 3
8	$\quad\quad\quad P(x)$	$\Leftrightarrow$ E, 4, 7
9	$\quad\quad\quad \exists x P(x)$	EG 8
10	$\quad\quad\quad \perp$	5, 9
11	$\quad\quad \exists x P(x)$	¬E, 5-16
12	$\quad\quad\quad \underline{\neg \exists x \neg P(x)}$ $\quad$ /∴ $\perp$	
13	$\quad\quad\quad \forall x P(x)$	QN 12
14	$\quad\quad\quad P(x)$	UI 13, 2
15	$\quad\quad\quad \neg P(y)$	$\Leftrightarrow$ E, 4, 14
16	$\quad\quad\quad \exists y \neg P(y)$	EG 15
17	$\quad\quad\quad \exists x \neg P(x)$	RDV 16
18	$\quad\quad\quad \perp$	12, 17
19	$\quad\quad \exists x \neg P(x)$	¬ E, 12-18
20	$\quad\quad (\exists x P(x) \wedge \exists x \neg P(x))$	∧ I, 11, 19
21	$\quad (\exists x P(x) \wedge \exists x \neg P(x))$	VEDC 3-20
22	$(\exists x P(x) \wedge \exists x \neg P(x))$	VUDC 2-21

#7 $\{\neg \exists x\, P(x) \vee \forall x\, P(x)\} \vdash \exists x \forall y\, (P(x) \Leftrightarrow P(y))$

1	$\neg \exists x\, P(x) \vee \forall x\, P(x)$ /∴ $\exists x \forall y\, (P(x) \Leftrightarrow P(y))$	
2	$\neg \exists x\, P(x)$	
3	$\exists x$	∃C
4	$\forall y$	UC
5	$P(x)$ /∴ $P(y)$	
6	$\exists x\, P(x)$	EG 5
7	$\bot$	2, 6
8	$P(y)$	⊥E
9	$P(y)$	
10	$\exists y\, P(y)$	EG 9
11	$\exists x\, P(x)$	RDV 10
12	$\bot$	2, 11
13	$P(x)$	⊥E
14	$(P(x) \Leftrightarrow P(y))$	⇔I, 5-8, 9-13
15	$\forall y\, (P(x) \Leftrightarrow P(y))$	UDC 4-14
16	$\exists x \forall y\, (P(x) \Leftrightarrow P(y))$	EDC 3-15
17	$\forall x\, P(x)$	
18	$\exists x$	EC
19	$\forall y$	UC
20	$P(x)$ /∴ $P(y)$	
21	$\forall x\, P(x)$	EG 20
22	$P(y)$	UI, 21, 19
23	$P(y)$	
24	$\forall x\, P(x)$	Imp 17
25	$P(x)$	UI 24, 18
26	$P(x) \Leftrightarrow P(y)$	⇔I, 20-22, 23-25
27	$\forall y\, (P(x) \Leftrightarrow P(y))$	UDC 19-26
28	$\exists x \forall y\, (P(x) \Leftrightarrow P(y))$	EDC 18-27
29	$\exists x \forall y\, (P(x) \Leftrightarrow P(y))$	∨E, 1, 2-16, 17-28

#8 $\phi \vdash \exists v \forall u \, \mathcal{Q}(v,u) \Rightarrow \forall u \exists v \, \mathcal{Q}(v,u)$ *

ϕ /∴ $\exists v \forall u \, \mathcal{Q}(v,u) \Rightarrow \forall u \exists v \, \mathcal{Q}(v,u)$

1	$\exists v \forall u \, \mathcal{Q}(v,u)$ /∴ $\forall u \exists v \, \mathcal{Q}(v,u)$	
2	$\forall u$	UC
3	$\exists v$	EC
4	$\forall u \, \mathcal{Q}(v,u)$	EC, 1
5	$\mathcal{Q}(v,u)$	UI, 4, 2
6	$\exists v \, \mathcal{Q}(v,u)$	EDC 3-5
7	$\forall u \exists v \, \mathcal{Q}(v,u)$	UDC 2-6
8	$\exists v \forall u \, \mathcal{Q}(v,u) \Rightarrow \forall u \exists v \, \mathcal{Q}(v,u)$	$\Rightarrow$I, 1-7

Let $L(x,y)$ stand for 'x loves y'. Then

$$\exists x \forall y \, L(x,y) \Rightarrow \forall y \exists x \, L(x,y)$$

says: 'If somebody loves everybody, then everybody is loved by somebody.'

But the converse is not true, or does not hold:

'If everybody is loved by somebody, then somebody loves everybody'

That is

$$\phi \nvdash \forall u \exists v \, \mathcal{Q}(v,u) \Rightarrow \exists v \forall u \, \mathcal{Q}(v,u)$$

The converse of * is not a theorem!

Some elementary semantics

$R(x)$: x is a raven
Let the univers of discourse consist of three individuals:

$$\mathcal{D} = \{a_1, a_2, a_3\}$$

Then we can establish the following semantic equivalences:

(a) $\forall x\, R(x) \models\!\!\!\dashv (R(a_1) \wedge R(a_2) \wedge R(a_3))$

If the left side is true then the right side is true, and conversely.

(b) $\exists x\, R(x) \models\!\!\!\dashv (R(a_1) \vee R(a_2) \vee R(a_3))$

If the left side is true then at least one disjunct of the right side must be true.

We can now show that the converse of * does not hold, from a semantic point of view. That is, we can show that

$$\begin{array}{l} 1.\ \forall y\, \exists x\, L(x,y) \\ \hline \therefore \exists x\, \forall y\, L(x,y) \end{array} \qquad (**)$$

is invalid from a semantic point of view.

If an argument is valid, then it is valid no matter how large the universe is.

Suppose $\mathcal{D} = \{a_1\}$, then (**) becomes

$$\frac{L(a_1, a_1)}{\therefore\ L(a_1, a_1)}$$

Clearly, if the conclusion is false then so is the premise. For $\mathcal{D} = \{a_1\}$, (**) is valid.

Suppose $\mathcal{D} = \{a_1, a_2\}$. Then premise

$\forall y \exists x\, L(x,y)$ is semantically equivalent to

$\vDash\!\dashv \forall y \{[L(a_1,y) \vee L(a_2,y)]\}$

$\vDash\!\dashv \{[L(a_1,a_1) \vee L(a_2,a_1)] \wedge [L(a_1,a_2) \vee L(a_2,a_2)]\}$

And the conclusion $\exists x \forall y\ L(x,y)$ is semantically equivalent to

$\vDash\!\dashv \exists x \{[L(x,a_1) \wedge L(x,a_2)]\}$

$\vDash\!\dashv \{[L(a_1,a_1) \wedge L(a_1,a_2)] \vee [L(a_2,a_1) \wedge L(a_2,a_2)]\}$

$$\begin{array}{l}
\quad\ \ \ (1) \qquad\qquad\quad (0) \qquad\quad \boxed{1} \qquad\quad (0) \qquad\qquad\quad (1) \\
1.\ \ \underline{(L(a_1,a_1) \vee L(a_2,a_1)) \wedge (L(a_1,a_2) \vee L(a_2,a_2))} \\
\therefore\ (L(a_1,a_1) \wedge L(a_1,a_2)) \wedge (L(a_2,a_1) \wedge L(a_2,a_2)) \\
\quad\ \ \ (1) \qquad\qquad\quad (0) \qquad\quad \boxed{0} \qquad\quad (0) \qquad\qquad\quad (1)
\end{array}$$

establishes that (**) is invalid.