Polyadic Quantification with Identity

Part III

Identity in English

Consider the sentence:

(A) There is problem that has exactly one solution

$P(x)$: x is a problem

$S(y,x)$: y is a solution of x

"exactly one solution" means

(a) "at least one solution and at most one solution"

(i) "x has at least one solution"

$$\exists y\, S(y,x)$$

(ii) "x has at most one solution"

$$\forall z_1 \forall z_2 \big((S(z_1,x) \land S(z_2,x)) \Rightarrow (z_1 = z_2) \big)$$

Therefore, (A) becomes

$$\exists x \big[P(x) \land \exists y\, S(y,x) \land \forall z_1 \forall z_2 \big((S(z_1,x) \land S(z_2,x)) \Rightarrow (z_1 = z_2) \big) \big]$$

Remark: To show there is exactly one term with a given property involves two steps:

(1) Show there is at least one such term

(2) Show that if there were two terms t_1, t_2, then $t_1 = t_2$

Statements of the following forms are very common in mathematics:

(1) There is one and only one u such that $\pi(u)$

(2) There is exactly one u such that $\pi(u)$

(3) There is a unique u such that $\pi(u)$

(4) There is at least one ~~one~~ u and at most one u such that $\pi(u)$

We translate each of them as:

$$\exists u\, \pi(u) \wedge \forall u \forall v \big((\pi(u) \wedge \pi(v)) \Rightarrow (u = v)\big) \stackrel{\text{def}}{=} \exists!\, u\, \pi(u)$$

where $\pi(v)$ is $\pi(v/u)$, and v is the first variable in the alphabetic list of variables that does not occur in $\pi(u)$.

Translation Examples

(a) There is at least two u's such that $\pi(u)$

$$\exists u_1 \exists u_2 \left(\pi(u_1) \wedge \pi(u_2) \wedge u_1 \neq u_2 \right)$$

(b) There exist at most two u's such that $\pi(u)$.

$$\forall u_1 \forall u_2 \forall u_3 \Big(\big(\pi(u_1) \wedge \pi(u_2) \wedge \pi(u_3)\big) \Rightarrow (u_1 = u_2) \vee (u_1 = u_3) \vee (u_2 = u_3)\Big)$$

(c) There exist exactly two u's s.t $\pi(u)$

Conjunction of (a) and (b)

Identity Introduction

1/ $\phi \vdash \exists x\,(x = x)$

$$
\begin{array}{l|l}
 & \underline{\phi} \\
 & \quad \begin{array}{|l} \underline{\exists x} \\ (x = x) \end{array} \qquad\qquad = I \\
 & \exists x\,(x = x)
\end{array}
$$

2/ $\phi \vdash \forall x\,(x = x)$

$$
\begin{array}{l|l}
 & \underline{\phi} \\
 & \quad \begin{array}{|l} \underline{\forall x} \\ x = x \end{array} \qquad\qquad = I \\
 & \forall x\,(x = x)
\end{array}
$$

3/ $\phi \vdash \exists x\,(x = a)$

$$
\begin{array}{l|l}
 & \underline{\phi} \\
 & a = a \qquad\qquad = I \\
 & \exists x\,(x = a)
\end{array}
$$

4/ $\phi \vdash \exists x\,(a = x)$

$$
\begin{array}{l|l}
 & \underline{\phi} \\
 & a = a \qquad\qquad = I \\
 & \exists x\,(a = x)
\end{array}
$$

Identity Introduction

6/ $\phi \vdash ((a=a) \Rightarrow \forall v\, \mathcal{Q}(v)) \Rightarrow \forall v\, \mathcal{Q}(v)$

1	$((a=a) \Rightarrow \forall v\, \mathcal{Q}(v))$	/∴ $\forall v\, \mathcal{Q}(v)$
2	$(a=a)$	$=I$
3	$\forall v\, \mathcal{Q}(v)$	$\Rightarrow E$
4	$((a=a) \Rightarrow \forall v\, \mathcal{Q}(v)) \Rightarrow \forall v\, \mathcal{Q}(v)$	$\Rightarrow I$

5/ $\phi \vdash \forall v\, \exists u\, (v = u)$

1	$\forall v$	
2	$(v = v)$	$=I$
3	$\exists u\, (v = u)$	EG 2
4	$\forall v\, \exists u\, (v = u)$	UDC 1–3

Identity Elimination

1. Suppose $2 < 2 + 2$
2. Suppose $1 + 1 = 2$

Then $1+1 < 2 + 2$

<u>Note</u>: We have <u>not</u> replaced all occurrences of 2 with 1+1 in this example.

Translating the above into formal notation:

$P(u, v, w) : u < (v + w)$

Now let 'a' represent '2'
'b' " '1 + 1'

Then the above example becomes

$$\begin{array}{|l} P(a, a, a) \\ \underline{b = a \qquad} \quad \therefore P(b, a, a) \\ \; \end{array}$$

Evidently, we want to provide in the rule of identity elimination not only for cases in which <u>all</u> occurrences of a term are replaced by occurrences of another, but for cases in which only <u>some</u> of these occurrences are ~~pr~~ replaced.

Examples:

$\mathcal{M}u$

$\quad\ldots$

i $\quad\quad \varphi(u,u) \vee \forall v\, \psi(v)$

j $\quad\quad u = a$

$\quad\quad\quad \varphi(a,a) \vee \forall v\, \psi(v) \qquad = E,\ i, j$

$\mathcal{M}u\ \mathcal{M}v$

$\quad\ldots$

i $\quad\quad \varphi(u,v,v) \Rightarrow \forall w\, \psi(w,v)$

j $\quad\quad u = v$

$\quad\quad\quad \varphi(u,v,u) \Rightarrow \forall w\, \psi(w,u) \qquad = E,\ i, j$

1 $a = b$

2 $b = c \quad / \therefore\ c = b$

3 $a = c \qquad = E, 1, 2$

4 $c = b \qquad = E, 1, 3$

$\varphi(a)$

$a = a \quad / \therefore\ \varphi(a)$

1 $\varphi(a)$

2 $\exists u((u = a) \wedge (u = b)) \quad / \therefore\ \varphi(b)$

3 $\quad \exists u$

4 $\quad (u = a) \wedge (u = b)$

5 $\quad u = a \qquad \wedge E$

6 $\quad \varphi(a) \qquad$ Import

7 $\quad \varphi(u) \qquad = E,\ 5, 6$

8 $\quad u = b \qquad \wedge E$

9 $\quad \varphi(b) \qquad = E,\ 7, 8$

10 $\varphi(b)$

$$\phi \vdash \forall u \exists v\, \mathcal{C}(v,u) \Rightarrow \forall u \exists v\, ((v = v) \wedge \mathcal{C}(v,u))$$

1	$\forall u \exists v\, \mathcal{C}(v,u)$	$\therefore\ \forall u \exists v\, ((v = v) \wedge \mathcal{C}(v,u))$
2	$\quad \forall u$	
3	$\quad \exists v\, \mathcal{C}(v,u)$	
4	$\quad\quad \exists v$	
5	$\quad\quad \mathcal{C}(v,u)$	
6	$\quad\quad (v = v)$	$= I$
7	$\quad\quad (v = v) \wedge \mathcal{C}(v,u)$	$\wedge I$
8	$\quad \exists v\, ((v = v) \wedge \mathcal{C}(v,u))$	$\exists DC$
9	$\forall u \exists v\, ((v = v) \wedge \mathcal{C}(v,u))$	
10	$\forall u \exists v\, \mathcal{C}(v,u) \Rightarrow \forall u \exists v\, ((v = v) \wedge \mathcal{C}(v,u))$	

1	$\exists v \forall u\, (\mathcal{C}(u) \Leftrightarrow (v = u))$	
2	$\mathcal{C}(a)$	
3	$\mathcal{C}(b)$	$\therefore\ (a = b)$
4	$\quad \exists v$	EC
5	$\quad \forall u\, (\mathcal{C}(u) \Leftrightarrow (v = u))$	$EC, 1$
6	$\quad \mathcal{C}(a) \Leftrightarrow (v = a)$	$UI, 5$
7	$\quad (v = a)$	$\Leftrightarrow E, 2, 6$
8	$\quad \mathcal{C}(b) \Leftrightarrow (v = b)$	$UI, 5$
9	$\quad (v = b)$	$\Leftrightarrow E, 3, 8$
10	$\quad (a = b)$	$= E, 7, 9$
11	$(a = b)$	

1	$\exists x \forall y (F(y) \Leftrightarrow (y = x))$	
2	$\forall x (F(x) \Leftrightarrow G(x))$	$/\therefore \exists x \forall y (G(y) \Leftrightarrow (y = x))$
3	$\exists x$	
4	$\forall y (F(y) \Leftrightarrow (y = x))$	
5	$\forall y$	$\forall x (F(x) \Leftrightarrow G(x))$ Imp. 2
6	$F(y) \Leftrightarrow G(y)$	
7	$G(y)$ $/\therefore (y = x)$	
8	$F(y)$	$\Leftrightarrow$E, 6, 7
9	$\forall y (F(y) \Leftrightarrow (y = x))$	Import 4
10	$F(y) \Leftrightarrow (y = x)$	UI, 9, 5
11	$y = x$	$\Rightarrow$E, 8, 10
12	$y = x$ $/\therefore G(y)$	
13	$\forall y (F(y) \Leftrightarrow (y = x))$	Import 4
14	$F(y) \Leftrightarrow (y = x)$	UI, 13, 5
15	$F(y)$	$\Leftrightarrow$ 12, 14
16	$\forall x (F(x) \Leftrightarrow G(x))$	Import 2
17	$F(y) \Leftrightarrow G(y)$	UI, 16, 5
18	$G(y)$	$\Leftrightarrow$E, 15, 17
19	$G(y) \Leftrightarrow (y = x)$	$\Leftrightarrow$I, 7–11, 12–18
20	$\forall y (G(y) \Leftrightarrow (y = x))$	UDC 5–19
21	$\exists x \forall y (G(y) \Leftrightarrow (y = x))$	EDC 3–20

1	$\exists x \forall y\, (y = x)$ $\quad$ $/\therefore \forall z\, (P(z) \Rightarrow \forall y\, P(y))$	
2	$\forall z$	
3	$P(z)$ $\quad$ $/\therefore \forall y\, P(y)$	
4	$\forall y$	
5	$\exists x \forall y\, (x = y)$	Import 1
6	$\exists x$	EC
7	$\forall y\, (x = y)$	EC 5
8	$P(z)$	Import 3
9	$x = z$	UI, 7, 2
10	$x = y$	UI, 7, 4
11	$y = z$	= E, 9, 10
12	$P(y)$	= E, 8, 11
13	$P(y)$	
14	$\forall y\, P(y)$	
15	$P(z) \Rightarrow \forall y\, P(y)$	
16	$\forall z\, (P(z) \Rightarrow \forall y\, P(y))$	

$\phi \vdash \exists x \exists y\, (x = y)$

	ϕ /∴ $\exists x \exists y\, (x = y)$	
1	$\neg \exists x \exists y\, (x = y)$ /∴ $\bot$	
2	$\forall x \forall y\, \neg (x = y)$	QN 1
3	$\forall x$	UC
4	$\forall y$	UC
5	$\forall x \forall y\, \neg (x = y)$	Imp. 2
6	$\forall y\, \neg (x = y)$	UI, 5, 3
7	$\neg (x = x)$	UI, 6, 3
8	$(x = x)$	= I
9	$\bot$	7, 8
10	$\bot$	
11	$\bot$	
12	$\exists x \exists y\, (x = y)$	$\neg$E, 1–11

Shorter way:

	ϕ /∴ $\exists x \exists y\, (x = y)$	
1	$(a = a)$	= I
2	$\exists y\, (a = y)$	EG 1
3	$\exists x \exists y\, (x = y)$	EG 2

26b

	ϕ /∴ $\forall x(P(x) \Leftrightarrow \exists y((x=y) \wedge P(y)))$	
1	$\forall x$	UC
2	$P(x)$ /∴ $\exists y((x=y) \wedge P(y))$	
3	$(x=x)$	$=I$
4	$((x=x) \wedge P(x))$	$\wedge I$, 2, 3
5	$\exists y((x=y) \wedge P(y))$	EG 4
6	$\exists y((x=y) \wedge P(y))$ /∴ $P(x)$	
7	$\exists y$	EC
8	$((x=y) \wedge P(y))$	EC 6
9	$(x=y)$	$\wedge E$, 8
10	$P(y)$	$\wedge E$, 8
11	$P(x)$	$=E$ 9, 10
12	$P(x)$	VEDQ 7-11
13	$P(x) \Leftrightarrow \exists y((x=y) \wedge P(y))$	$\Leftrightarrow I$, 2-5, 6-12
14	$\forall x(P(x) \Leftrightarrow \exists y((x=y) \wedge P(y)))$	UDC 1-13

26c

1	$\forall x \forall y\, (R(x,y) \Rightarrow (x=y))$	
2	$F(a)$	
3	$\neg F(b)$	$/\therefore \neg R(a,b)$
4	$\quad R(a,b) \quad /\therefore \bot$	
5	$\quad \forall x \forall y\, (R(x,y) \Rightarrow (x=y))$	Imp. 1
6	$\quad \forall y\, (R(a,y) \Rightarrow (a=y))$	UI, 5
7	$\quad (R(a,b) \Rightarrow (a=b))$	UI, 6
8	$\quad a = b$	$\Rightarrow$E, 4, 7
9	$\quad \neg F(b)$	Imp. 3
10	$\quad \neg F(a)$	$=$E, 8, 9
11	$\quad \bot$	2, 10
12	$\neg R(a,b)$	$\neg$I 4–11

1	$\forall x\, ((x=a) \vee (x=b))$	
2	$\neg F(a)$ $/\therefore\ \exists x F(x) \Rightarrow F(b)$	
3	$\mid\ \exists x F(x)$ $/\therefore\ F(b)$	
4	$\mid\ \mid\ \exists x$	EC
5	$\mid\ \mid\ ((x=a) \vee (x=b))$	EC 1
6	$\mid\ \mid\ F(x)$	EC 3
7	$\mid\ \mid\ \mid\ (x=a)$	
8	$\mid\ \mid\ \mid\ F(a)$	$=E$, 6, 7
9	$\mid\ \mid\ \mid\ F(a) \vee F(b)$	$\vee I$, 8
10	$\mid\ \mid\ \mid\ F(b)$	DS 2, 9
11	$\mid\ \mid\ \mid\ (x=b)$	
12	$\mid\ \mid\ \mid\ F(b)$	$=E$ 6, 11
13	$\mid\ \mid\ F(b)$	$\vee E$, 5, 7–10, 11–12
14	$\mid\ F(b)$	VEDC 4–13
15	$\exists x F(x) \Rightarrow F(b)$	$\Rightarrow I$, 3–14

	$\varnothing$ /∴ $\forall x \forall y ((x = y) \Rightarrow (F(x) \Leftrightarrow F(y)))$	
1	$\forall x$	UC
2	$\forall y$	UC
3	$(x = y)$ /∴ $(F(x) \Leftrightarrow F(y))$	
4	$F(x)$ /∴ $F(y)$	
5	$F(y)$	$=$E, 3,4
6	$F(y)$	
7	$F(x)$	$=$E, 3,6
8	$(F(x) \Leftrightarrow F(y))$	$\Leftrightarrow$I, 4-5, 6-7
9	$(x = y) \Rightarrow (F(x) \Rightarrow F(y))$	$\Rightarrow$I, 3-8
10	$\forall y ((x = y) \Rightarrow (F(x) \Rightarrow F(y)))$	UDC 2-9
11	$\forall x \forall y ((x = y) \Rightarrow (F(x) \Rightarrow F(y)))$	UDC 1-10

Identity Relation has three properties

(1) Reflexivity

Anything is identical to itself. It is this which underlies $= I$. Thus

$$\left|\ a = a \qquad = I\right. \qquad\qquad \left|\ \underline{\forall v}\ \ v = v \qquad = I.\right.$$

(2) Symmetry:

$\emptyset \vdash (a = b) \Rightarrow (b = a)$

1	$\quad\mid\ \underline{(a=b)}$ /∴ $b = a$	
2	$\quad\mid\ a = a$	$= I$
3	$\quad\mid\ b = a$	$= E$, 1, 2
4	$(a = b) \Rightarrow (b = a)$	

$\emptyset \vdash \forall u \forall v((u = v) \Rightarrow (v = u))$

1	$\quad\mid\ \underline{\forall u}$	
2	$\quad\mid\quad\mid\ \underline{\forall v}$	
3	$\quad\mid\quad\mid\quad\mid\ \underline{u = v}$ /∴ $v = u$	
4	$\quad\mid\quad\mid\quad\mid\ u = u$	$= I$
5	$\quad\mid\quad\mid\quad\mid\ v = u$	$= E$, 3, 4
6	$\quad\mid\quad\mid\ (u = v) \Rightarrow (v = u)$	
7	$\quad\mid\ \forall v((u = v) \Rightarrow (v = u))$	
8	$\forall u \forall v((u = v) \Rightarrow (v = u))$	

(3) Transitivity

$$\phi \vdash ((a = b) \wedge (b = c)) \Rightarrow (a = c)$$

1	$(a = b) \wedge (b = c)$	$/ \therefore\ a = c$
2	$a = b$	
3	$b = c$	
4	$a = a$	$= I$
5	$b = a$	$= E$ 2, 4
6	$c = a$	$= E$, 3, 5
7	$c = c$	$= I$
8	$a = c$	$= E$ 6, 7
9	$((a = b) \wedge (b = c)) \Rightarrow (a = c)$	

$$\phi \vdash \forall u\, \forall v\, \forall w\, [((u = v) \wedge (v = w)) \Rightarrow (u = w)]$$

1	$\forall u$	
2	$\forall v$	
3	$\forall w$	
4	$(u = v) \wedge (v = w)$	$/ \therefore\ (u = w)$
5	$u = v$	$\wedge E$
6	$v = w$	$\wedge E$
7	$u = \cancel{u}$	$= I$
8	$v = u$	$= E$ 5, 7
9	$w = u$	$= E$, 6, 8
10	$w = w$	$= I$
11	$u = w$	$= E$, 9, 10
12	$((u = v) \wedge (v = w)) \Rightarrow (u = w$	
13	$\forall u\, \forall v\, \forall w\, [\quad]$	

Short cut, doing several step at once.

$$\phi \vdash \mathcal{Q}(a) \Leftrightarrow \exists x\,((x=a) \wedge \mathcal{Q}(x))$$

1	$\mathcal{Q}(a) \quad /\therefore\ \exists x((x=a) \wedge \mathcal{Q}(x))$	
2	$a=a$	$=I$
3	$(a=a) \wedge \mathcal{Q}(a)$	$\wedge I$
4	$\exists x((x=a) \wedge \mathcal{Q}(x)$	EG 3
5	$\exists x((x=a) \wedge \mathcal{Q}(x) \quad /\therefore\ \mathcal{Q}(a)$	
6	$\exists x$	
7	$x=a \wedge \mathcal{Q}(x)$	
8	$x=a$	$\wedge E$
9	$\mathcal{Q}(x)$	$\wedge E$
10	$\mathcal{Q}(a)$	$=E$, 8, 9
11	$\mathcal{Q}(a)$	
12	$\mathcal{Q}(a) \Leftrightarrow \exists x((x=a) \wedge \mathcal{Q}(x))$	$\Leftrightarrow I$

16 $\{\exists x \exists y \forall z (z = x \vee z = y), \mathcal{Q}(a) \wedge \mathcal{Q}(b), \neg(a = b)\} \vdash \forall x\, \mathcal{Q}(x)$

1	$\exists x \exists y \forall z (z = x \vee z = y)$	
2	$\mathcal{Q}(a) \wedge \mathcal{Q}(b)$	
3	$\neg(a = b)$	$/ \therefore \forall x\, \mathcal{Q}(x)$
4	$\neg \forall z (z = a \vee z = b)$	
5	$\exists z (\neg z = a \wedge \neg z = b)$	
6	$\exists x$	
7	$\exists y$	
8	$\forall z (z = x \vee z = y)$	
9	$\exists z (\neg z = a \wedge \neg z = b)$	
10	$\exists z$	
11	$z = x \vee z = y$	
12	$\neg(z = a) \wedge \neg(z = b)$	
13	$\neg(z = a)$	
14	$\neg(z = b)$	
15	$(a = x \vee a = y)$	UI 8
16	$(b = x \vee b = y)$	UI 8
17	$a = x$	
18	$b = x$	
19	$a = b$	$=$E 17, 18
20	$\bot$	3, 19
21	$b = y$	
22	$\neg(z = a)$	R 13
23	$a = x$	R 17
24	$\neg(z = x)$	$=$E 22, 23
25	$\neg(z = b)$	R 14
26	$\neg(z = y)$	$=$E 21, 25
27	$\neg(z = x) \wedge \neg(z = y)$	$\wedge$I 24, 26
28	$\neg(z = x) \vee (z = y))$	DeM.
29	$\bot$	11, 28
30	$\bot$	$\vee$E, 16, 18–20, 21–28
31	$a = y$	
32	$b = y$	
33	$a = b$	$=$E, 31, 32

1 2 3 4 5 6 7

1 2 3 4 5 6 7

34 $\perp$, 33, 3

35 $b = x$

36 $\neg(z = a)$ R 13

37 $a = y$ R 31

38 $\neg(z = y)$ $=$ E,

39 $\neg(z = b)$ R 14

40 $\neg(z = x)$ $=$E 35, 39

41 $\neg(z = x) \wedge \neg(z = y)$ $\wedge$ I 40, 38

42 $\neg((z = x) \vee (z = y))$

43 $\perp$ 11, 42

44 $\perp$ $\vee$E, 16, 32-34, 35-43

45 $\perp$ $\vee$E, 15, 17-30, 31-~~44~~

46 $\perp$

47 $\perp$

48 $\perp$

49 $\forall z\,((z = a) \vee (z = b))$

50 $\forall x\,(x = a \vee x = b)$ R DV 49

51 $\forall x$

52 $(x = a) \vee (x = b)$

53 $x = a$

54 $\mathcal{Q}(a)$ $\wedge$E, 2, I

55 $\mathcal{Q}(x)$ $=$E 53, 54

56 $x = b$

57 $\mathcal{Q}(b)$ $\wedge$E, 2, I

58 $\mathcal{Q}(x)$ $=$E 53, 57

59 $\mathcal{Q}(x)$ $\vee$E, 52, 53-55, 56-58

60 $\forall x\,\mathcal{Q}(x)$

Remark

Hypothesis (1) states that there exist at most two things. The use of 'a' and 'b' ~~states~~ indicates that their names are 'a' and 'b'. The last hypothesis states that the things are distinct. This suggests that one should first prove $\forall x (x = a \vee x = b)$ from H1 and H3. This essentially happens from 4 – 50. One then makes use of H2. This occurs from 51 — 60.

Part 4

1/ All cars are useful. Therefore, everyone who has a car has something useful.

$P(u)$: u is a car
$Q(u)$: u is useful
$R(u)$: u is a person
$S(u, v)$: u has v

$\mathcal{D} = \{$ all physical objects $\}$

$$\{\forall x\,(P(x) \Rightarrow Q(x))\} \vdash \forall y\,\forall x\,(((R(y) \wedge S(y,x)) \wedge P(x)) \Rightarrow \exists z\,(Q(z) \wedge S(y,z)))$$

1	$\forall x\,(P(x) \Rightarrow Q(x))$ /∴ $\forall y\,\forall x\,(((R(y) \wedge S(y,x)) \wedge P(x)) \Rightarrow \exists z\,(Q(z) \wedge S(y,z)))$	
2	$\forall y$	
3	$\forall x$	
4	$((R(y) \wedge S(y,x)) \wedge P(x))$ /∴ $\exists z\,(Q(z) \wedge S(y,z))$	
5	$P(x)$	
6	$\forall x\,(P(x) \Rightarrow Q(x))$	
7	$P(x) \Rightarrow Q(x)$	UI.
8	$Q(x)$	$\Rightarrow$E, 5, 7
9	$R(y) \wedge S(y,x)$	$\wedge$E, 4
10	$S(y, x)$	$\wedge$E, 9
11	$Q(x) \wedge S(y,x)$	$\wedge$I, 8, 10
12	$\exists z\,(Q(z) \wedge S(y,z))$	EG.
13	$((R(y) \wedge S(y,x)) \wedge P(x)) \Rightarrow \exists x\,(Q(z) \wedge S(y,z))$	
14	$\forall x\,(\qquad)$	
15	$\forall y\,\forall x\,(\qquad)$	

2. Pegasus is Pegasus. Therefore Pegasus is something.

Pegasus : a

$\mathcal{D}$ = { mythical animals }

$a = a \vdash \exists x (x = a)$

$$\begin{array}{|l}
a = a \\
\hline
\exists x (x = a) \qquad \text{E6.}
\end{array}$$

4. Any cause of itself is a cause of anything else. Therefore any cause of itself is a cause of everything.

$P(u, v)$: u causes v

$\mathcal{D} = \{\text{all things}\}$

$\{\forall x\,(P(x,x) \Rightarrow \forall y\,(\neg(x=y) \Rightarrow P(x,y)))\}$

$\vdash \forall x\,(P(x,x) \Rightarrow \forall y\, P(x,y))$

1	$\forall x\,(P(x,x) \Rightarrow \forall y(\neg(x=y) \Rightarrow P(x,y)))$	
2	$\forall x$	
3	$P(x,x)$ /∴ $\forall y\, P(x,y)$	
4	$\forall y$	
5	$P(x,x)$	Imp 3
6	$\forall x\,(P(x,x) \Rightarrow \forall y\,(\neg(x=y) \Rightarrow P(x,y)))$	R 1
7	$P(x,x) \Rightarrow \forall y\,(\neg(x=y) \Rightarrow P(x,y))$	UI 6,
8	$\forall y\,(\neg(x=y) \Rightarrow P(x,y))$	$\Rightarrow$E, 5, 7
9	$\neg(x=y) \Rightarrow P(x,y)$	UI 8,
10	$\neg P(x,y)$ /∴ $\perp$	
11	$x=y$	
12	$P(x,x)$	R 3
13	$P(x,y)$	=E 11, 12
14	$\neg P(x,y)$	R 10
15	$\perp$	
16	$\neg(x=y)$	$\neg$I 11–15
17	$\neg(x=y) \Rightarrow P(x,y)$	Imp 9
18	$P(x,y)$	
19	$\perp$	
20	$P(x,y)$	
21	$\forall y\, P(x,y)$	
22	$(P(x,x) \Rightarrow \forall y\, P(x,y))$	
23	$\forall x\,(P(x,x) \Rightarrow \forall y\, P(x,y))$	

12 Only Smith and the guard at the gate knew the password. Someone who knew the password stole the gun. Therefore either Smith or the guard at the gate stole the gun.

$K(u)$: u knew the password
$S(u)$: u stole the gun
s : Smith
g : guard

$\mathcal{D}$ = {people}

$$\{\forall x(K(x) \Rightarrow ((x=s) \vee (x=g))), \exists x(K(x) \wedge S(x))\} \vdash (S(s) \vee S(g))$$

1	$\forall x(K(x) \Rightarrow ((\cancel{x}=s) \vee (x=g)))$	
2	$\exists x(K(x) \wedge S(x)$ $\quad /\therefore\ S(s) \vee S(g)$	
3	$\quad \exists x$	
4	$\quad K(x) \Rightarrow ((x=s) \vee (x=g))$	
5	$\quad K(x) \wedge S(x)$	
6	$\quad K(x)$	
7	$\quad S(x)$	
8	$\quad (x=s) \vee (x=g)$	
9	$\qquad x=s$	
10	$\qquad S(x)$	R 7
11	$\qquad S(s)$	=E, 9, 10
12	$\qquad S(s) \vee S(g)$	∨I 11
13	$\qquad x=g$	
14	$\qquad S(x)$	R 7
15	$\qquad S(g)$	=E, 13, 14
16	$\qquad S(s) \vee S(g)$	∨I 15
17	$\quad S(s) \vee S(g)$	∨E, 8, 9-12, 13-16
18	$S(s) \vee S(g)$	

14/ There is a line connecting any two distinct points. P is a point. Therefore, if any point is distinct from P, there is a line connecting it with P.

$P(u)$: u is a point

$Q(u)$: u is a line

$R(u, v, w)$: u connects v and w.

$\mathcal{D}$ = { points and lines is some fixed space of points and lines }.

Show :

$$\{\forall x \, \forall y [((P(x) \wedge P(y) \wedge \neg(x=y)) \Rightarrow \exists z \, (Q(z) \wedge R(z,x,y))], \; P(a)\} \vdash$$
$$\forall x [(P(x) \wedge \neg(a=x)) \Rightarrow \exists z (Q(z) \wedge R(z,a,x))]$$

Q4 contd

1+ 1 | $\forall x \forall y ((P(x) \land P(y) \land \neg(x = y)) \Rightarrow \exists z (Q(z) \land R(z, x, y)))$

$P(a)$ /∴ $\forall x ((P(x) \land \neg(a = x)) \Rightarrow \exists x (Q(z) \land R(z, a, x)))$

$\forall x$

$P(x) \land \neg(a = x)$

$\forall x \forall y ((P(x) \land P(y) \land \neg(x = y)) \Rightarrow \exists z (Q(z) \land R(z, x, y)))$

$\forall y ((P(a) \land P(y) \land \neg(a = y)) \Rightarrow \exists z (Q(z) \land R(z, a, y)))$

$((P(a) \land P(x) \land \neg(a = x)) \Rightarrow \exists z (Q(z) \land R(z, a, x)))$

$P(a)$ R

$P(x) \land \neg(a = x)$ R

$(P(a) \land P(x) \land \neg(a = x))$

$\exists z (Q(z) \land R(z, a, x)))$

$(P(x) \land \neg(a = x)) \Rightarrow \exists z (Q(z) \land R(z, a, x)))$

$\forall x ((P(x) \land \neg(a = x)) \Rightarrow \exists x (Q(z) \land R(z, a, x)))$