

Predicate Logic Derivations I

Quantifier Negation

1) $\forall v Q(v) \vdash \neg \exists v \neg Q(v)$

1		$\forall v Q(v)$	\therefore	$\neg \exists v \neg Q(v)$	
2			$\exists v \neg Q(v)$	\therefore	\perp
3				$\exists v$	EC
4				$Q(v)$	EC 1
5				$\neg Q(v)$	EC 2
6				\perp	\perp I, 4, 5
7				\perp	
8		$\neg \exists v \neg Q(v)$			

1		$\neg \exists v \neg Q(v)$	\therefore	$\forall v Q(v)$	
2			$\forall v$	$\therefore Q(v)$	UC
3				$\neg Q(v)$	
4				$\exists v \neg Q(v)$	EG 2
5				\perp	\perp I, 1, 4
6				$Q(v)$	\neg E, 3-5
7		$\forall v Q(v)$			UDC 2-6

2) $\exists v Q(v) \vdash \neg \forall v \neg Q(v)$

1		$\exists v Q(v)$	\therefore	$\neg \forall v \neg Q(v)$	
2			$\forall v \neg Q(v)$	\therefore	\perp
3				$\exists v$	EC
4				$Q(v)$	EC 1
5				$\neg Q(v)$	EC 2
6				\perp	\perp I, 4, 5
7				\perp	
8		$\neg \forall v \neg Q(v)$			\neg I, 2-7

+	1	$\neg \forall v \neg Q(v) \therefore \exists v Q(v)$	
	2	$\neg \exists v Q(v) \therefore \perp$	
	3	$\forall v$	UC
	4	$Q(v) \therefore \perp$	
	5	$\exists v Q(v)$	EG
	6	\perp	$\perp I, 2, 5$
	7	$\neg Q(v)$	$\neg I, 4-6$
	8	$\forall v \neg Q(v)$	UDC, 3-7
	9	\perp	$\perp I, 1, 8$
	10	$\exists v Q(v)$	$\neg E, 2-9$

3)	1	$\neg \forall v Q(v) \therefore \exists v \neg Q(v)$	
	2	$\neg \exists v \neg Q(v) \therefore \perp$	
+	3	$\forall v$	UC
	4	$\neg Q(v) \therefore \perp$	
	5	$\exists v \neg Q(v)$	EG 4
	6	\perp	$\perp I, 2, 5$
	7	$Q(v)$	$\neg I, 4-6$
	8	$\forall v Q(v)$	UDC, 3-7
	9	\perp	$\perp I, 1, 8$
	10	$\exists v \neg Q(v)$	$\neg E, 2-9$

+	1	$\exists v \neg Q(v) \therefore \neg \forall v Q(v)$	
	2	$\forall v Q(v) \therefore \perp$	
	3	$\exists v$	EC
	4	$\neg Q(v)$	EC, 1
	5	$Q(v)$	EC, 2
	6	\perp	$\perp I, 4, 5$
	7	\perp	
	8	$\neg \forall v Q(v)$	$\neg I, 2-7$

$\therefore \neg \forall v Q(v) \vdash \neg \exists v \neg Q(v)$

+	1	$\neg \forall v \neg Q(v) \therefore \exists v Q(v)$	
	2	$\neg \exists v Q(v) \therefore \perp$	
	3	$\forall v$	UC
	4	$Q(v) \therefore \perp$	
	5	$\exists v Q(v)$	EG
	6	\perp	$\perp I, 2, 5$
	7	$\neg Q(v)$	$\neg I, 4-6$
	8	$\forall v \neg Q(v)$	UDC, 3-7
	9	\perp	$\perp I, 1, 8$
	10	$\exists v Q(v)$	$\neg E, 2-9$

3)	1	$\neg \forall v Q(v) \therefore \exists v \neg Q(v)$	
	2	$\neg \exists v \neg Q(v) \therefore \perp$	
+	3	$\forall v$	UC
	4	$\neg Q(v) \therefore \perp$	
	5	$\exists v \neg Q(v)$	EG 4
	6	\perp	$\perp I, 2, 5$
	7	$Q(v)$	$\neg I, 4-6$
	8	$\forall v Q(v)$	UDC, 3-7
	9	\perp	$\perp I, 1, 8$
	10	$\exists v \neg Q(v)$	$\neg E, 2-9$

+	1	$\exists v \neg Q(v) \therefore \neg \forall v Q(v)$	
	2	$\forall v Q(v) \therefore \perp$	
	3	$\exists v$	EC
	4	$\neg Q(v)$	EC, 1
	5	$Q(v)$	EC, 2
	6	\perp	$\perp I, 4, 5$
	7	\perp	
	8	$\neg \forall v Q(v)$	$\neg I, 2-7$

$\therefore \neg \forall v Q(v) \vdash \neg \exists v \neg Q(v)$

4) $\neg \exists x Q(x) \vdash \forall x \neg Q(x)$

1	$\neg \exists x Q(x) \quad / \therefore \quad \forall x \neg Q(x)$	
2	$\forall x$	
3	$Q(x) \quad / \therefore \quad \perp$	
4	$\exists x Q(x)$	EG 3
5	\perp	$\perp I, 1, 4$
6	$\neg Q(x)$	$\neg I, 3-5$
7	$\forall x \neg Q(x)$	UDC 2-6

1	$\forall x \neg Q(x) \quad / \therefore \quad \neg \exists x Q(x)$	
2	$\exists x Q(x) \quad / \therefore \quad \perp$	
3	$\exists x$	EC
4	$\neg Q(x)$	EC, 1
5	$Q(x)$	EC, 2
6	\perp	$\perp I, 4, 5$
7	\perp	
8	$\neg \exists x Q(x)$	$\neg I, 2-7$

$$T1: \phi \vdash \forall v (Q(v) \Rightarrow \psi(v)) \Rightarrow (\forall v Q(v) \Rightarrow \forall v \psi(v))$$

1	$\forall v (Q(v) \Rightarrow \psi(v)) \quad \therefore (\forall v Q(v) \Rightarrow \forall v \psi(v))$	
2	$\forall v Q(v) \quad \therefore \forall v \psi(v)$	
3	$\forall v$	UC
4	$Q(v) \Rightarrow \psi(v)$	UC 1
5	$Q(v)$	UC 2
6	$\psi(v)$	$\Rightarrow E, 4, 5$
7	$\forall v \psi(v)$	UDC, 3-6
8	$\forall v Q(v) \Rightarrow \forall v \psi(v)$	$\Rightarrow I, 2-7$
9	$\forall v (Q(v) \Rightarrow \psi(v)) \Rightarrow (\forall v Q(v) \Rightarrow \forall v \psi(v)) \Rightarrow I 1-8$	

$$T2: \phi \vdash \forall v (Q(v) \Rightarrow \psi(v)) \Rightarrow (\exists v Q(v) \Rightarrow \exists v \psi(v))$$

1	$\forall v (Q(v) \Rightarrow \psi(v)) \quad \therefore \exists v Q(v) \Rightarrow \exists v \psi(v)$	
2	$\exists v Q(v) \quad \therefore \exists v \psi(v)$	
3	$\exists v$	EC
4	$Q(v) \Rightarrow \psi(v)$	EC 1
5	$Q(v)$	EC 2
6	$\psi(v)$	$\Rightarrow E$
7	$\exists v \psi(v)$	EDC 3-6
8	$\exists v Q(v) \Rightarrow \exists v \psi(v)$	$\Rightarrow I, 2-7$
9	$\forall v (Q(v) \Rightarrow \psi(v)) \Rightarrow (\exists v Q(v) \Rightarrow \exists v \psi(v)) \Rightarrow I, 1-8$	

Remark: The intuitive plausibility of T1 and T2 can be seen in this:

$F(x)$: x is a person
 $G(x)$: x is mortal

- Then T1 and T2 can be translated into:
- (a) If all persons are mortal, then if everything is a person, then everything is mortal
 - (b) If all persons are mortal, then if something is a person, then something is mortal.

QD 2 (Quantifier Distribution)

$$\exists v (Q(v) \vee \psi(v)) \vdash \exists v Q(v) \vee \exists v \psi(v)$$

<p>⊢</p> <p>1 <u>$\exists v (Q(v) \vee \psi(v))$</u> $\therefore \exists v Q(v) \vee \exists v \psi(v)$</p>		
2	<u>$\exists v$</u>	EC
3	$Q(v) \vee \psi(v)$	EC, 1
4	<u>$Q(v)$</u>	
5	$\exists v Q(v)$	
6	$\exists v Q(v) \vee \exists v \psi(v)$	$\vee I, 5$
7	<u>$\psi(v)$</u>	
8	$\exists v \psi(v)$	
9	$\exists v Q(v) \vee \exists v \psi(v)$	$\vee I, 8$
10	$\exists v Q(v) \vee \exists v \psi(v)$	$\vee E, 3, 4-6, 7-9$
11	$\exists v (\exists v Q(v) \vee \exists v \psi(v))$	$\exists v Q, 2-10$
12	$\exists v Q(v) \vee \exists v \psi(v)$	$\exists v D Q, 11$

<p>⊢</p> <p>1 <u>$\exists v Q(v) \vee \exists v \psi(v)$</u> $\therefore \exists v (Q(v) \vee \psi(v))$</p>		
2	<u>$\exists v Q(v)$</u>	
3	<u>$\exists v$</u>	EC
4	$Q(v)$	EC 2
5	$Q(v) \vee \psi(v)$	$\vee I, 4$
6	$\exists v (Q(v) \vee \psi(v))$	EDC, 3-5
7	<u>$\exists v \psi(v)$</u>	
8	<u>$\exists v$</u>	EC
9	$\psi(v)$	EC, 7
10	$Q(v) \vee \psi(v)$	$\vee I, 9$
11	$\exists v (Q(v) \vee \psi(v))$	EDC, 8-10
12	$\exists v (Q(v) \vee \psi(v))$	$\vee E, 1, 2-6, 7-11$

T4: $\phi \vdash (\forall v Q(v) \vee \forall v \psi(v)) \Rightarrow \forall v (Q(v) \vee \psi(v))$

1	$\forall v Q(v) \vee \forall v \psi(v)$	$\therefore \forall v (Q(v) \vee \psi(v))$	
2	$\forall v Q(v)$		
3	$\forall v$		UC
4	$Q(v)$		UC, 2
5	$Q(v) \vee \psi(v)$		$\vee I$, 4
6	$\forall v (Q(v) \vee \psi(v))$		UDC, 3-5
7	$\forall v \psi(v)$		
8	$\forall v$		UC
9	$\psi(v)$		UC, 7
10	$Q(v) \vee \psi(v)$		$\vee I$, 9
11	$\forall v (Q(v) \vee \psi(v))$		UDC, 8-10
12	$\forall v (Q(v) \vee \psi(v))$		$\vee E$, 1, 2-6, 7-11

T5: $\phi \vdash (\exists v Q(v) \Rightarrow \exists v \psi(v)) \Rightarrow \exists v (Q(v) \Rightarrow \psi(v))$

1	$\exists v Q(v) \Rightarrow \exists v \psi(v)$	$\therefore \exists v (Q(v) \Rightarrow \psi(v))$	
2	$\neg \exists v Q(v) \vee \exists v \psi(v)$		Sentential Logic 1
3	$\neg \exists v Q(v)$		
4	$\forall v \neg Q(v)$		QN 3 (quantifier)
5	$\neg Q(a)$		UI, 4 (negation)
6	$\neg Q(a) \vee \psi(a)$		$\vee I$, 5
7	$\exists v (\neg Q(v) \vee \psi(v))$		EG, 6
8	$\exists v \psi(v)$		
9	$\exists v$		EC
10	$\psi(v)$		EC, 8
11	$\neg Q(v) \vee \psi(v)$		$\vee I$, 10
12	$\exists v (\neg Q(v) \vee \psi(v))$		EG, 11
13	$\exists v (\neg Q(v) \vee \psi(v))$		$\vee E$, 2, 3-7, 8-12
14	$(\exists v Q(v) \Rightarrow \exists v \psi(v)) \Rightarrow \exists v (Q(v) \Rightarrow \psi(v))$		$\Rightarrow I$, 1-13

13a $\exists v (Q(v) \Rightarrow \psi(v))$

T6: $\phi \vdash (\forall v Q(v) \Rightarrow \forall v \psi(v)) \Rightarrow \exists v (Q(v) \Rightarrow \psi(v))$

1	<u>$\forall v Q(v) \Rightarrow \forall v \psi(v)$</u> $\therefore \exists v (Q(v) \Rightarrow \psi(v))$															
2	$\neg \forall v Q(v) \vee \forall v \psi(v)$	SL (Impl 1)														
3	$\exists v \neg Q(v) \vee \forall v \psi(v)$															
4	<table border="0" style="border-left: 1px solid black; padding-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><u>$\exists v \neg Q(v)$</u></td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <table border="0" style="border-left: 1px solid black; padding-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><u>$\exists v$</u></td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\neg Q(v)$</td> <td style="padding-left: 20px;">EC</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\neg Q(v) \vee \psi(v)$</td> <td style="padding-left: 20px;">EC 4</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Q(v) \Rightarrow \psi(v)$</td> <td style="padding-left: 20px;">$\vee I, 6$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\exists v (Q(v) \Rightarrow \psi(v))$</td> <td style="padding-left: 20px;">Impl, 7</td> </tr> </table> </td> <td style="padding-left: 20px;">EDC, 5-8</td> </tr> </table>	<u>$\exists v \neg Q(v)$</u>		<table border="0" style="border-left: 1px solid black; padding-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><u>$\exists v$</u></td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\neg Q(v)$</td> <td style="padding-left: 20px;">EC</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\neg Q(v) \vee \psi(v)$</td> <td style="padding-left: 20px;">EC 4</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Q(v) \Rightarrow \psi(v)$</td> <td style="padding-left: 20px;">$\vee I, 6$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\exists v (Q(v) \Rightarrow \psi(v))$</td> <td style="padding-left: 20px;">Impl, 7</td> </tr> </table>	<u>$\exists v$</u>		$\neg Q(v)$	EC	$\neg Q(v) \vee \psi(v)$	EC 4	$Q(v) \Rightarrow \psi(v)$	$\vee I, 6$	$\exists v (Q(v) \Rightarrow \psi(v))$	Impl, 7	EDC, 5-8	
<u>$\exists v \neg Q(v)$</u>																
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<u>$\exists v$</u>																
$\neg Q(v)$	EC															
$\neg Q(v) \vee \psi(v)$	EC 4															
$Q(v) \Rightarrow \psi(v)$	$\vee I, 6$															
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5																
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8																
9																
10	<u>$\forall v \psi(v)$</u>															
11	$\psi(a)$	UI, 10														
12	$\neg Q(a) \vee \psi(a)$	$\vee I, 11$														
13	$Q(a) \Rightarrow \psi(a)$	Impl. 12														
14	$\exists v (Q(v) \Rightarrow \psi(v))$	EG, 13														
15	$\exists v (Q(v) \Rightarrow \psi(v))$	$\vee E, 3, 4-9, 10-14$														
16	$(\forall v Q(v) \Rightarrow \forall v \psi(v)) \Rightarrow \exists v (Q(v) \Rightarrow \psi(v))$															

T7: $\phi \vdash \forall v (Q(v) \Leftrightarrow \psi(v)) \Rightarrow (\forall v Q(v) \Leftrightarrow \forall v \psi(v))$

1	<u>$\forall v (Q(v) \Leftrightarrow \psi(v))$</u> $\therefore (\forall v Q(v) \Leftrightarrow \forall v \psi(v))$											
2	<u>$\forall v Q(v)$</u> $\therefore \forall v \psi(v)$											
3	<table border="0" style="border-left: 1px solid black; padding-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><u>$\forall v$</u></td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Q(v) \Leftrightarrow \psi(v)$</td> <td style="padding-left: 20px;">UC</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Q(v)$</td> <td style="padding-left: 20px;">UC, 1</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\psi(v)$</td> <td style="padding-left: 20px;">UC, 2</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\psi(v)$</td> <td style="padding-left: 20px;">$\Leftrightarrow E, 4, 5$</td> </tr> </table>	<u>$\forall v$</u>		$Q(v) \Leftrightarrow \psi(v)$	UC	$Q(v)$	UC, 1	$\psi(v)$	UC, 2	$\psi(v)$	$\Leftrightarrow E, 4, 5$	UDC, 3-6
<u>$\forall v$</u>												
$Q(v) \Leftrightarrow \psi(v)$	UC											
$Q(v)$	UC, 1											
$\psi(v)$	UC, 2											
$\psi(v)$	$\Leftrightarrow E, 4, 5$											
4												
5												
6												
7	$\forall v \psi(v)$											
8	<u>$\forall v \psi(v)$</u> $\therefore \forall v Q(v)$											
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<u>$\forall v$</u>												
$Q(v) \Leftrightarrow \psi(v)$	UC											
$\psi(v)$	UC, 1											
$Q(v)$	UC, 2											
$Q(v)$	$\Leftrightarrow E, 10, 11$											
10												
11												
12												
13	$\forall v Q(v)$											
14	$\forall v Q(v) \Leftrightarrow \forall v \psi(v)$ ($\Leftrightarrow I, 2-7, 8-13$)											
15	$\forall v (Q(v) \Leftrightarrow \psi(v)) \Rightarrow (\forall v Q(v) \Leftrightarrow \forall v \psi(v))$ ($\Rightarrow I, 1-14$)											

T8: $\phi \vdash \forall v(Q(v) \Leftrightarrow \psi(v)) \Rightarrow (\exists v Q(v) \Leftrightarrow \exists v \psi(v))$

1	$\forall v(Q(v) \Leftrightarrow \psi(v)) \quad \therefore \exists v Q(v) \Leftrightarrow \exists v \psi(v)$
2	$\exists v Q(v) \quad \therefore \exists v \psi(v)$
3	$\exists v$
4	$Q(v) \Leftrightarrow \psi(v)$
5	$Q(v)$
6	$\psi(v)$
7	$\exists v \psi(v)$
8	$\exists v \psi(v)$
9	$\exists v$
10	$Q(v) \Leftrightarrow \psi(v)$
11	$\psi(v)$
12	$Q(v)$
13	$\exists v Q(v)$
14	$\exists v Q(v) \Leftrightarrow \exists v \psi(v) \quad \Leftrightarrow I$
15	$\forall v(Q(v) \Leftrightarrow \psi(v)) \Rightarrow (\exists v Q(v) \Leftrightarrow \exists v \psi(v)) \Rightarrow I$

T9: $\phi \vdash \forall v(Q(v) \Leftrightarrow \alpha) \Rightarrow (\forall v Q(v) \Leftrightarrow \alpha)$

1	$\forall v(Q(v) \Leftrightarrow \alpha) \quad \therefore (\forall v Q(v) \Leftrightarrow \alpha)$
2	$\forall v Q(v) \quad \therefore \alpha$
3	$\forall v$
4	$Q(v) \Leftrightarrow \alpha$
5	$Q(v)$
6	α
7	$\forall v \alpha \quad UVQC, 3-6$
8	$\alpha \quad UVDA, 7 \quad \text{Not } v \in FV[\alpha]$
9	α
10	$\forall v \quad UC$
11	$Q(v) \Leftrightarrow \alpha \quad UC, 1$
12	$\alpha \quad \text{Reit, 9}$
13	$Q(v)$
14	$\forall v Q(v)$
15	$\forall v Q(v) \Leftrightarrow \alpha \quad \Leftrightarrow I, 2-8, 9-14$
16	$\forall v(Q(v) \Leftrightarrow \alpha) \Rightarrow \forall v Q(v) \Leftrightarrow \alpha \quad \Rightarrow I, 1-15$

T10: $\phi \vdash \forall v (Q(v) \Leftrightarrow \alpha) \Rightarrow (\exists v Q(v) \Leftrightarrow \alpha)$

1	$\forall v (Q(v) \Leftrightarrow \alpha)$	$\% (\exists v Q(v) \Leftrightarrow \alpha)$	
2		$\exists v Q(v)$	$\therefore \alpha$
3		$\exists v$	
4		$Q(v) \Leftrightarrow \alpha$	
5		$Q(v)$	
6		α	
7		$\exists v \alpha$	
8		α	
9		α	$\therefore \exists v Q(v)$
10		$\exists v$	
11		α	
12		$Q(v) \Leftrightarrow \alpha$	
13		$Q(v)$	
14		$\exists v Q(v)$	
15	$\exists v Q(v) \Leftrightarrow \alpha$		$(\Rightarrow) I, 2-8, 9-14$
16	$\forall v (Q(v) \Leftrightarrow \alpha) \Rightarrow (\exists v Q(v) \Leftrightarrow \alpha)$		$\Rightarrow I, 1-15.$

Reit 9, allowed because $v \notin FV[\alpha]$

T11: $\phi \vdash (\exists v Q(v) \Leftrightarrow \alpha) \Rightarrow \exists v (Q(v) \Leftrightarrow \alpha)$

1	$(\exists v Q(v) \Leftrightarrow \alpha)$	$\therefore \exists v (Q(v) \Leftrightarrow \alpha)$	
2		$\neg \exists v (Q(v) \Leftrightarrow \alpha)$	$\therefore \perp$
3		$\forall v \neg (Q(v) \Leftrightarrow \alpha)$	$QN \int \neg(Q \Leftrightarrow \neg Q) \vdash -$
4		$\forall v (Q(v) \Leftrightarrow \neg \alpha)$	$SL \left\{ (Q \Leftrightarrow \neg Q) \right.$
5		$(\exists v Q(v) \Leftrightarrow \neg \alpha)$	$T10$
6		$\neg (\exists v Q(v) \Leftrightarrow \alpha)$	SL
7		\perp	$\perp I, 1, 6$
8	$\exists v (Q(v) \Leftrightarrow \alpha)$		$\neg E, 2-7$
9	$(\exists v Q(v) \Leftrightarrow \alpha) \Rightarrow \exists v (Q(v) \Leftrightarrow \alpha)$		$\Rightarrow I$

T12: $\phi \vdash (\forall v (Q(v) \Leftrightarrow \alpha) \Rightarrow \exists v (Q(v) \Leftrightarrow \alpha))$

1	$(\forall v (Q(v) \Leftrightarrow \alpha) \Leftrightarrow \exists v (Q(v) \Leftrightarrow \alpha)) \quad \therefore \perp$	
2	$\neg \exists v (Q(v) \Leftrightarrow \alpha) \quad \therefore \perp$	
3	$\forall v \neg (Q(v) \Leftrightarrow \alpha)$	Q N
4	$\forall v (Q(v) \Leftrightarrow \neg \alpha)$	SL 3
5	$\forall v Q(v) \Leftrightarrow \neg \alpha$	T 9, 4
6	$\neg (\forall v Q(v) \Leftrightarrow \alpha)$	SL
7	\perp	\perp I, 1, 6
8	$\exists v (Q(v) \Leftrightarrow \alpha)$	\neg E, 2-7

T13: $\phi \vdash \exists v (\exists v Q(v) \Rightarrow Q(v))$

1	$\neg \exists v (\exists v Q(v) \Rightarrow Q(v)) \quad \therefore \perp$	
2	$\forall v \neg (\exists v Q(v) \Rightarrow Q(v))$	Q N
3	$\forall v \neg (\neg \exists v Q(v) \vee Q(v))$	Impl 2
4	$\forall v (\neg \neg \exists v Q(v) \wedge \neg Q(v))$	De Morgan
5	$\forall v (\exists v Q(v) \wedge \neg Q(v))$	DN 4
6	$(\forall v \exists v Q(v) \wedge \forall v \neg Q(v))$	Q D
7	$\forall v \exists v Q(v)$	\wedge E
8	$\exists v Q(v)$	
9	$\forall v \neg Q(v)$	\wedge E, 6
10	$\exists v$	EC
11	$Q(v)$	EC 8
12	$\neg Q(v)$	EC 9
13	\perp	
14	\perp	
15	$\exists v (\exists v Q(v) \Rightarrow Q(v))$	Vacuous Quant 7

T 14: $\phi \vdash \exists v (Q(v) \Rightarrow \forall v Q(v))$

1	$\neg \exists v (Q(v) \Rightarrow \forall v Q(v)) \quad \perp$	\perp				
2	$\forall v (Q(v) \wedge \neg \forall v Q(v))$	QN, Impl., DeM				
3	$\forall v (Q(v) \wedge \exists v \neg Q(v))$	QN				
4	<table border="0" style="border-left: 1px solid black; border-right: 1px solid black; margin-left: 10px;"> <tr> <td style="padding-right: 5px;">$\forall v$</td> <td></td> </tr> <tr> <td style="padding-right: 5px;">$Q(v) \wedge \exists v \neg Q(v)$</td> <td></td> </tr> </table>	$\forall v$		$Q(v) \wedge \exists v \neg Q(v)$		UC
$\forall v$						
$Q(v) \wedge \exists v \neg Q(v)$						
5	$Q(v) \wedge \exists v \neg Q(v)$	UC, 3				
6	$Q(v)$	$\wedge E, 5$				
7	$\exists v \neg Q(v)$	$\wedge E, 5$				
8	$\forall v Q(v)$	UDC, 6				
9	$\forall v \exists v \neg Q(v)$	UDC, 7				
10	$\exists v \neg Q(v)$	UVDC, 9				
11	<table border="0" style="border-left: 1px solid black; border-right: 1px solid black; margin-left: 10px;"> <tr> <td style="padding-right: 5px;">$\exists v$</td> <td></td> </tr> <tr> <td style="padding-right: 5px;">$Q(v)$</td> <td></td> </tr> </table>	$\exists v$		$Q(v)$		EC
$\exists v$						
$Q(v)$						
12	$Q(v)$	EC, 8				
13	$\neg Q(v)$	EC, 10				
14	\perp	$\perp I, 12, 13$				
15	\perp	$\perp I, 12, 13$				
16	$\exists v (Q(v) \Rightarrow \forall v Q(v))$	$\neg E, 1-15$				